Basics of Hamiltonian Mechanics

Liouville's Theorem

Poincare Recurrence

→ Basics of Hamiltonian Mechanics

– Why?

L.E.: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$ → 2nd order eqn. for gen. coords.

$$\left\{ \frac{d}{dt}\Big(\underset{\substack{\downarrow \\ \text{generalized} \\ \text{momentum}}}{p_i}\Big) = \frac{\partial L}{\partial q} \right\}$$

H.E.: $$\dot{p} = -\partial H/\partial q$$ → 2 first order equations

$$\dot{q} = \partial H/\partial p$$

→ coordinates and momenta on equal footing in fact interchangeable ...

H.E. very useful for phase space descriptions, formulations.

N.B.: History:

- Lagrange, 1756 (France)
  ⇒ minimization
- Hamilton, 1823 (Ireland)
  ⇒ outgrowth of ray tracing using Huygens' Principle

- Formulation: Legendre transformation $\dot{q} \to p$

In general $L = L(q, \dot{q})$

n.b. t is parameter

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}$$

$$= \dot{p}\, dq + p\, d\dot{q}$$

$$d(p\dot{q}) = p\, d\dot{q} + \dot{q}\, dp$$

$$dL = \dot{p}\, dq + d(p\dot{q}) - \dot{q}\, dp$$

$$d(p\dot{q} - L) = -\dot{p}\, dq + \dot{q}\, dp$$

$$= dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

n.b. - $H = H(q, p)$
- Legendre transform via construction

so, equating

$$\dot{p} = -\frac{\partial H}{\partial q} \, , \quad \dot{q} = \frac{\partial H}{\partial p} \quad \Bigg\} \quad \text{Hamiltonian EOMs.}$$

→ Hamiltonian is function of generalized coordinates and momenta. Very impt.

* → to construct Hamiltonian formulation, need *not* have conservative system. Need *only* be able to invert:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \, , \quad \text{to solve } \dot{q}_i \text{ in terms } p_i \ldots$$

N.B.: Conservation?

— in Lagrangian mechanics,

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L \quad \Rightarrow \text{linked time translation symmetry}$$

— $\partial_t L = 0 \Rightarrow \dot{E} = 0$.

Now, in Hamiltonian Mechanics:

$$\frac{dH}{dt} = \frac{\partial}{\partial t} H + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$$

$$= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q}\left(\frac{-\partial H}{\partial p}\right) + \frac{\partial H}{\partial p}\left(\frac{\partial H}{\partial q}\right)$$

$$= \partial H / \partial t.$$

Thus, if no explicit time dependence, energy conserved $\left(E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L,\ H = p\dot{q} - L\right)$, and $H = \text{const.}$ (equiv. to $\partial L/\partial t = 0$ in Lagrangian formulation)

→ Constructing Hamiltonians.

→ trivial.

Particle moves in $U(r, \theta, \phi)$. Construct Hamiltonian?

$$L = T - U$$

$$= \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 - U$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

so $$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) - U$$

Now, $$H = \underline{p} \cdot \underline{\dot{q}} - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

and

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

so $$\dot{r} = p_r / m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

so $$\dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \partial L / \partial \dot{\phi} = m r^2 \sin^2\theta \, \dot{\phi}$$

so $$\dot{\phi} = p_\phi / m r^2 \sin^2\theta$$

Now need eliminate generalized velocities

$$H = p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$-\left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta}\right) - U$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + U$$

and Hamiltonian EOMs follow.

→ off-beat

$$L(q,\dot{q}) = e^{\dot{q}}$$

$$H = p\dot{q} - L$$

Now $p = \frac{\partial L}{\partial \dot{q}} = e^{\dot{q}}$ ⇒ $\dot{q} = \ln p$

$$H = p\ln p - p$$

$$\dot{q} = \ln p \, , \quad \dot{p} = 0$$

→ time dependent

$$L = \frac{1}{2}\sigma(q,t)\dot{q}^2 + F(q,t)\dot{q} - V(q,t)$$

Again: → $H = p\dot{q} - L$

→ $p = \frac{\partial L}{\partial \dot{q}}$

$$p = \sigma(q,t)\dot{q} + F(q,t)$$

$$\dot{q} = (p - F(q,t))/\sigma(q,t)$$

⇒

$$H = p\left(\frac{p-F}{\sigma}\right) - \frac{\sigma}{2}\frac{(p-F)^2}{\sigma^2} - F\frac{(p-F)}{\sigma} + V$$

$$= \frac{(p-F)^2}{2\sigma} + V$$

and H EoMs follow.

N.B. $\left.\begin{matrix} F = F(q,t) \\ \sigma = \sigma(q,t) \end{matrix}\right\}$ here

→ in general, Hamiltonian formulation requires invertibility of generalized velocities in terms of generalized momenta.

i.e. need solve $p_i = \frac{\partial L}{\partial \dot{q}_i}$ for $\dot{q}_i(p, q)$

to eliminate $\dot{q}_i$!

Generally,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

locally,

$$dp_i = d\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} d\dot{q}_j$$

⇒

$$dp_i = A_{ij}\, d\dot{q}_j \quad , \quad A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

$d\dot{q}_j = A^{-1}_{ij}\, dp_i$ and can solve and eliminate.

Obviously,
- A_{ij} must be invertible
- $\det A_{ij} \neq 0$ required!

If det $A_{ij} = 0$ ⇒ special constraint exists, requiring treatment by Dirac brackets, instead Poisson brackets. Via that approach, can still formulate Hamiltonian.

Ex. → Give an example of a system for which a conventional Hamiltonian cannot be formulated. Explain why.

Non-trivial example: (cf Dirac lectures, '64)

Consider a charged particle in x-y plane, in magnetic field $B_0 \hat{z}$.
→ strong field limit

$$\Rightarrow \quad L = \frac{1}{2} m v^2 + \frac{Z}{c} \underline{v} \cdot \underline{A} - U$$

$$\underline{A} = \frac{B_0}{2} \hat{z} \times \underline{r} = \frac{B_0}{2} (x \hat{y} - y \hat{x})$$

so can re-scale as:

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{Z B_0}{2c} (x \dot{y} - y \dot{x}) - \frac{Z B_0}{2c} U$$

and rescale by m to obtain:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{eB_0}{2mc}(x\dot{y} - y\dot{x}) - \frac{eB_0}{2mc}U(x,y)$$

Now $\eta \equiv eB_0/2mc = \frac{\Omega_{cyc}}{2}$.

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \eta(x\dot{y} - y\dot{x}) - \eta U(x,y)$$

Now, consider $\eta >> d/dt$, so

$$\eta(x\dot{y} - y\dot{x}) >> \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

N.B. - strong field limit, i.e. drop kinetic energy.
- here Lagrangian linear in velocity ⇒ obvious difficulty in inversion!

i.e. $L \cong \eta(x\dot{y} - y\dot{x}) - \eta U(x,y)$

L EOMs: $\frac{d}{dt}(-y) = -\partial U/\partial x$

$\frac{d}{dt}(x) = -\partial U/\partial y$

Now, for Hamiltonian:

$$p_x = -\eta y = \partial L/\partial \dot{x}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \eta x$$

and <u>no</u> inversion of $\dot{x}, \dot{y}$ in terms p_x, p_y possible!! → special feature is strong field constraint

Furthur[illegible]:

$$H = p_x \dot{x} + p_y \dot{y} - L$$

$$= \dot{x}(-\eta y) + \dot{y}(\eta x) - \eta(x\dot{y} - y\dot{x}) + \eta U$$

$$= \eta U$$

(akin G.C. Plasma)

Momenta drop out!.

What is the problem here?

- Lagrangian <u>linear</u> in <u>V</u>

⇔

- Coordinates (q's) and momenta (p's) <u>not</u> independent.

→ Need attack by adding constraint (aka Lagrange multiplier) to usual story.

TBC.

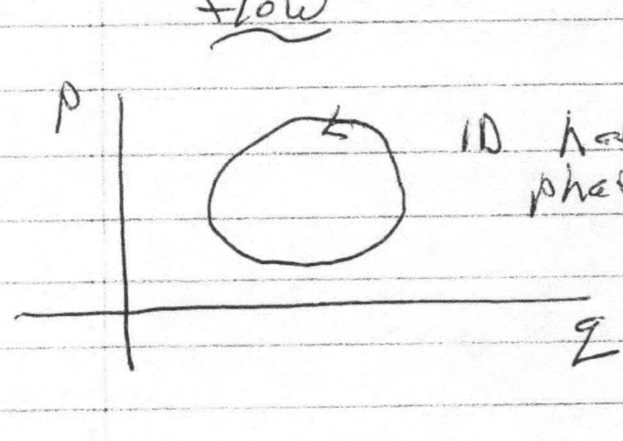
→ Using Hamiltonians

- by treating q, p symmetrically Hamiltonians are natural variables for phase space description of dynamics

i.e. → replaces 2nd order Lagrange equation with 2 first order Hamilton eqn.

→ natural for describing phase space flow

i.e.



1D harmonic oscillator phase portrait.

i.e. for describing phase space dynamics, need:

- phase space density $\rho(q,p)$ and its evolution

  i.e. $\underset{\downarrow}{f(q,p)} \leftrightarrow \rho(q,p)$

  distribution function

$$\langle E_K \rangle = \int d^{3N}p \int d^{3N}q \; \frac{p^2}{2m} f(q,p) \Big/ V_\Gamma$$

- understanding of nature of the flow.

Now, if $V_\Gamma = (\underset{\downarrow}{\dot{q}}, \dot{p})$

phase space flow (2ND dim. vector)

then $\underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = 0 \Rightarrow$ flow is incompressible

i.e.

$$\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p} = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial}{\partial p}\left(\frac{-\partial H}{\partial q}\right)$$

$$= 0$$

consequence only of Hamiltonian structure!

⇒ generic to Hamiltonian structure, phase space flow is incompressible.

⇒ phase space density conserved along particle trajectories

i.e. <u>for particles not created or destroyed,</u>

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}_{\Gamma} \cdot (\rho \underline{V}_{\Gamma}) = 0$$

{phase space continuity

i.e.

$$\frac{\partial \rho}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial \underline{q}_i} \cdot (\dot{\underline{q}}_i \rho) + \frac{\partial}{\partial \underline{p}_i} \cdot (\dot{\underline{p}}_i \rho) \right\} = 0$$

so

$$\frac{\partial \rho}{\partial t} + \underline{V}_{\Gamma} \cdot \underline{\nabla}_{\Gamma} \rho + \rho \underline{\nabla} \cdot \underline{V}_{\Gamma} = 0$$

and

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\dot{\underline{q}}_i \cdot \frac{\partial}{\partial \underline{q}_i} \rho + \dot{\underline{p}}_i \cdot \frac{\partial}{\partial \underline{p}_i} \rho \right)$$

$$+ \sum_i \rho \left(\frac{\partial}{\partial \underline{q}_i} \cdot \dot{\underline{q}}_i + \frac{\partial}{\partial \underline{p}_i} \cdot \dot{\underline{p}}_i \right) = 0$$

for Hamiltonian system:

$\underline{\nabla} \cdot \underline{V}_\Pi = 0 \longleftrightarrow$ phase space flow incompressible

Phase volume conserved !

$\boxed{\text{Liouville's Thm.}}$

so

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{V}_\Pi \cdot \underline{\nabla}_\Pi \rho = 0}$$

⇒ {Phase space density conserved along particle trajectories.

⇒ Locally conserved phase space density.

n.b. for N particle system

$\rho = \rho(\underline{p}_1, \underline{q}_1, \cdots \underline{p}_N, \underline{q}_N) \rightarrow$ N body distribution fctn.

$$\partial_t \rho + \sum_{i=1}^{N} \left(\dot{\underline{q}}_i \cdot \frac{\partial}{\partial \underline{q}_i} + \dot{\underline{p}}_i \cdot \frac{\partial}{\partial \underline{p}_i} \right) \rho = 0$$

if dilute, etc. can derive:
Boltzmann Eqn. (via BBGKY hierarchy)

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f + \underline{a} \cdot \frac{\partial f}{\partial \underline{v}} = C(f,f)$$

collision operator
(→ 2 body interaction)

if collisionless:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f + \underline{a} \cdot \nabla_v f = 0$$

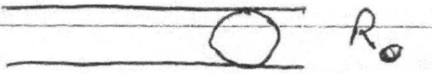
Vlasov equation

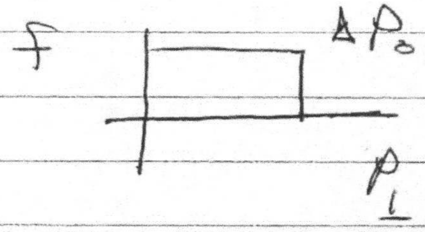
Example:

Consider a particle beam, with transverse momentum dispersion $\Delta_\perp P_0$, and radius R_0. Comment on what will happen if attempt to focus to $R_1 <$ [illegible] R_0.

- Consider beam as Hamiltonian system.

ie





Key:
- phase space volume conserved
- conservative irrelevant/ no a priori connection of conservative dynamics and Hamiltonian structure

$$V_{ph}\Big|_{\text{before focus}} = V_{ph}\Big|_{\text{after focus}}$$

$$\pi R_0^2 \, \pi (\Delta P_{\perp 0})^2 = \pi R_1^3 \, \pi (\Delta P_{\perp 1})^2$$

$$\Rightarrow \quad \Delta P_{\perp 1} = \frac{R_0}{R_1} \Delta P_{\perp 0}$$

so dispersion increases, to compensate reduction in spatial focal point region.

$\Rightarrow$ inefficient.

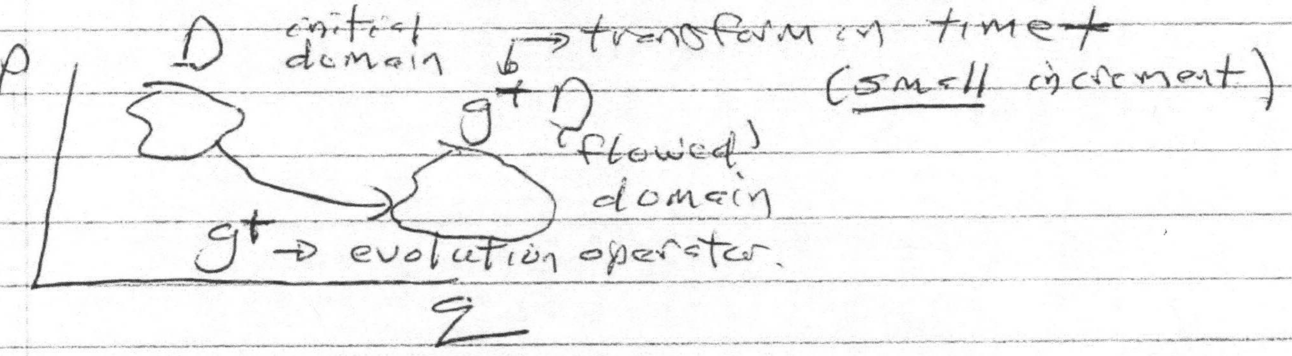
→ Poincaré Recurrence Theorem

another take on phase space flow, Liouville's theorem:

Define phase flow g^t: transformation s/t

$\underline{p}(0), \underline{q}(0) \rightarrow \underline{p}(t), \underline{q}(t)$ along Hamiltonian trajectories.

i.e.



Now:

- Hamiltonian eqns ~~define~~ constitute define autonomous system

i.e. $\underline{\dot{x}} = \underline{F}(\underline{x})$

$\underline{V}_H =$ [redacted] $\begin{pmatrix} \partial H/\partial \underline{p} \\ -\partial H/\partial \underline{q} \end{pmatrix} = \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix}$

then, for small increment:

$$\underline{g}^t(\underline{x}) = \underline{x} + \underline{f}(\underline{x})\, t + O(t^2)$$

so then phase volume at t:

$$V_\Gamma(t) = \int\limits_{\underset{\substack{\uparrow \\ \text{initial} \\ \text{domain}}}{D(0)}} d\underline{x} \left| \overset{\substack{\text{Jacobian of transform} \\ \downarrow}}{\frac{\partial \underline{\underline{x}}'}{\partial \underline{x}}} \right|$$

$$= \int_{D(0)} d\underline{x} \;\; \det \left| \frac{\partial \underline{g}^t(\underline{x})}{\partial \underline{x}} \right|$$

Now

$$\frac{\partial \underline{g}^t(\underline{x})}{\partial \underline{x}} = \underline{\underline{I}} + \frac{\partial \underline{f}}{\partial \underline{x}}\, t + O(t^2)$$

but now use identity (small t):

$$\det\left(\underline{\underline{I}} + \underset{+ \cdots}{\underline{\underline{A}}\, t}\right) = 1 + t \operatorname{tr} \underline{\underline{A}} + \ldots$$

so

$$V(t) = \int_{D(0)} d^3x \left[1 + t \, \mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{x}} \right] + O(t^2) \right]$$

but $\mathrm{tr} \dfrac{\partial f}{\partial \underline{x}} = \underline{\nabla} \cdot \underline{f}$

from $\underline{V}_\Gamma = \underline{f}$, $\quad \underline{\nabla} \cdot \underline{f} = \underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = 0$

$\downarrow$

f is phase space flow velocity

so, to t^2,

$V(t) = V(0)$ $\Rightarrow$ phase volume conserved.

as expected.

$\Rightarrow$ no attractors in Hamiltonian mechanics i.e. no asymptotically stable positions, cycles.

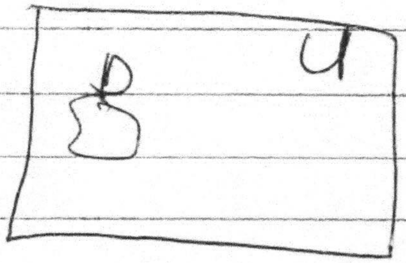
so come to:

Poincare Recurrence Theorem

- fundamental to ergodic theory
- inspiration for F. Nietche...

$\Rightarrow$ -loosely, " what goes around, comes around, arbitrarily closely", for bounded Hamiltonian system...

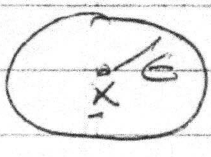
state;



$U \equiv$ system universe, bounded.

g^t Hamiltonian, so volume preserving.

For any $\underline{x}$ in U, can define $B(\underline{x}, \epsilon)$



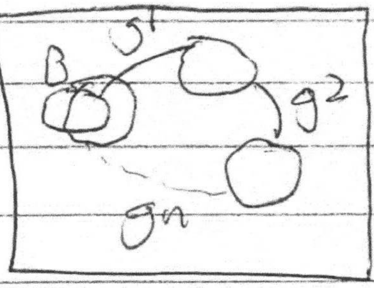
ball in phase space around pt $\underline{x}$ $(\underline{P}, \underline{\Sigma})$ of radius ϵ

then $\exists$ $\underline{x}' \in B(\underline{x})$ s/t

$$g^n(\underline{x}') \in B(\underline{x})$$

i.e. {there is a point in the ϵ-ball of $\underline{x}$ such that n iterations of evolution operator yield a point also in the ϵ ball.

i.e. {point recurs, arbitrarily closely" i.e.



consider $g^n(B)$

if each g^i disjoint

$\lim_{n \to \infty} \bigcup g^n \to \infty$, but U bounded

$\Rightarrow$ contradiction

so

$g^k(B) \cap g^l(B) \neq \phi$ intersection of arbitrary iterates not empty.

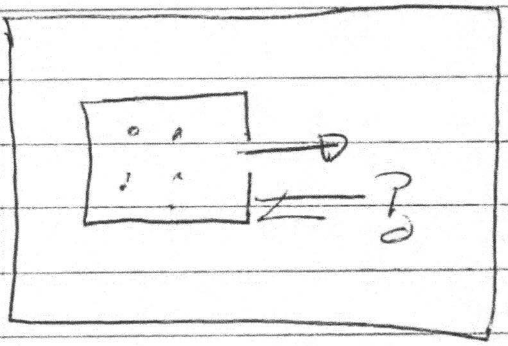
$\Rightarrow$ $g^{k-l}(B) \cap B \neq \phi$

so $\exists$ some $x' \in g^{k-l}(B) \cap B$

so there is some $\underline{x}'$ arbitrarily close to $\underline{x}$

QED

Implications:

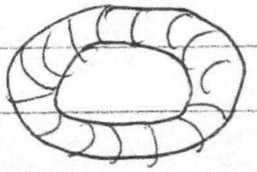
- 

box with particles.

⇒ particles escape thru hole

⇒ eventually, will go back in

but may be a while.

- if torus 

$\dot{\varphi}_1 = \alpha_1$

$\dot{\varphi}_2 = \alpha_2$

α_1 / α_2 irrational)

then if $g^t(\varphi_1, \varphi_2) \to (\varphi_1 + \alpha_1 t, \varphi_2 + \alpha_2 t)$

α_1/α_2 irrational ⇒ winding fills torus.

comes arbitrarily close ----

→ Poincare Recurrence – FAQ's :

- refs:
  - V.I. Arnold, "Mathematical Methods of Classical Mechanics"
  - S. Chandrasekhar "Stochastic Problems in Physics and Astronomy"
    Rev. Mod. Phys. 15, 1 (1943), online
  - G. Zaslavsky "Hamiltonian Chaos and Fractional Dynamics".

- Why Care? (apart from interest)
  - ergodic theory

i.e. $$\underset{\substack{\downarrow \\ \text{ensemble} \\ \text{avg.}}}{\langle A \rangle_{\text{ensemble}}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt\, A(t)$$

ensemble avg. ⟺ time average

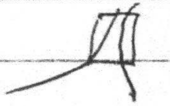
points :
- $\underset{\substack{\uparrow \\ \text{range of } \epsilon}}{B(x, \epsilon)}$ ⟹ trajectory [redacted] returns arbitrarily closely to i.c.

- any ensemble avg ⟹ partition ⟹

⇒ coarse graining $\Delta P, \Delta E$

- time average guaranteed to fill the space, as will find $\Pi_1 - \Pi_2 <$

$\sqrt{(\Delta p)^2+(\Delta \varepsilon)^2}$

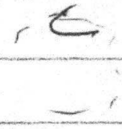


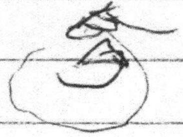
- what of harmonic oscillator ?

n.b. - oscillator ≠ limit cycle

$\downarrow$ closed trajectory but <u>not</u> attractor

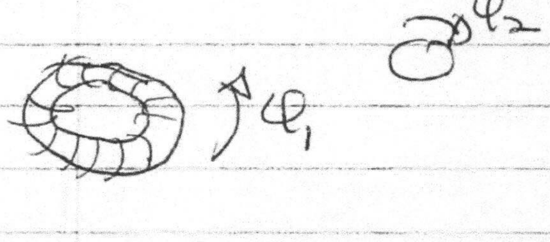
$\downarrow$ attractor





✗ - closed, periodic trajectories are generally the exception (though surely possible)

i.e. consider toroidal surface



$\dot{\varphi}_1 = \alpha_1$,

$\varphi_2 = \alpha_2$

α_1/α_2 → rational ⇒ closed cycle
⇒ curve

α_1/α_2 → irrational → $(g^t)^n$ winding fills surface, an some iteration comes *arbitarily* *close* to initial point.
⇒ surface

[illegible] n.b. # irrationals >> # rationals.

- *time* for recurrence is ***long***.